Exact boson mapping of the reduced BCS pairing Hamiltonian

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Abstract

An exact boson mapping of the reduced BCS (equal strength) pairing Hamiltonian is considered. In the mapping, fermion pair operators are mapped exactly to the corresponding bosons. The image of the mapping results in a Bose-Hubbard model with level dependent hopping. Though the resultant Bose-Hubbard Hamiltonian is non-Hermitian, all eigenvalues are real when $Uk/t < 1$, where k is the total number of bosons. When $U/t = 1$, a part of spectrum of the Bose-Hubbard Hamiltonian corresponds exactly to the whole spectrum of the reduced BCS pairing Hamiltonian.

Keywords: reduced BCS paring Hamiltonian, boson mapping, the Bose-Hubbard model

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Pairing is one of important residue interactions in many areas of physics, especially in the studies of superconductors,[1, 2] nuclear systems,[3] metallic clusters,[4, 5] and liquids.[6] The limitations of the Bardeen-Cooper-Schrieffer (BCS) and Hartree-Fock-Bogolyubov (HFB) methods[1,7] for finding approximate solutions of finite nuclear systems and nanoscale metallic grains are well understood.[4, 8] Fortunately, the reduced BCS (equal strength) pairing model was proved to be exactly solvable following Richardson's early work[9, 10] and studies based on the Gaudin algebraic Bethe ansatz method,[11] which has received a lot of attention recently.[12, 13]

On the other hand, there is also a long history in search for appropriate boson mapping methods or boson expansions for nuclear many-body systems,[14] which can also be carried out for other many-fermion systems. It is well known that in most circumstances pairs of fermions exhibit boson-like behavior. In such approaches, the degrees of freedom of fermion pairs are directly replaced by exact boson degrees of freedom. These methods are potentially helpful in describing collective motion in terms of boson degrees of freedom to avoid usual difficult fermionic formulation since boson operators have their counterparts in classical canonical variables, and thus provide a direct link between microscopic nuclear models and phenomenological collective models. A lot of attention has been paid[14] especially after the success of the interacting boson model for nuclei.[15]

The purpose of this letter is to report an exact boson mapping of the reduced BCS pairing Hamiltonian. When m single fermions occupy the j_1, j_2,$\cdots$, j_m levels, respectively, the reduced BCS pairing Hamiltonian for deformed nuclear system is given by

$$\hat{H}_{\rm BCS}/G = \sum_{\tau=1}^{m}(\epsilon_{j_\tau}/G)(c^\dagger_{j_\tau\uparrow}c_{j_\tau\uparrow} + c^\dagger_{j_\tau\downarrow}c_{j_\tau\downarrow}) + \sum_{j}{}'\eta_j\hat{k}_j - \sum_{i,j}{}'S^+_iS^-_j, \tag{1}$$

where $c^\dagger_{j\sigma}$ ($c_{j\sigma}$) are fermion creation (annihilation) operators, $S^+_j = c^\dagger_{j\uparrow}c^\dagger_{j\downarrow}$ and $S^-_j = c_{j\downarrow}c_{j\uparrow}$ are pair creation (annihilation) operators, $\hat{k}_j = (c^\dagger_{j\uparrow}c_{j\uparrow} + c^\dagger_{j\downarrow}c_{j\downarrow})/2$, ϵ_j are single-particle energies taken from any deformed mean-field theory, $G > 0$ is the equal strength pairing parameter, $\eta_j = 2\epsilon_j/G$, and the summation sign with prime indicates that the sum is restricted to levels other than those occupied by the single fermions.

Because solutions of $m \neq 0$ cases are basically similar to those of seniority zero case, in the following we only consider the case with $m = 0$. For k-particle excitation, the wavefunction of (1) in this case with p levels considered can be written as[9, 10]

$$|k;\xi\rangle = S^+(E^{(\xi)}_1)S^+(E^{(\xi)}_2)\cdots S^+(E^{(\xi)}_k)|0\rangle, \tag{2}$$

where $|0\rangle$ is the pairing vacuum state satisfying $S_j^+|0\rangle = 0$ for $1 \le j \le p$,

$$S^+(E_\mu^{(\xi)}) = \sum_{j=1}^{p} \frac{1}{\eta_j - E_\mu^{(\xi)}} S_j^+, \tag{3}$$

with the corresponing eigen-energy $E_k^{(\xi)} = G \sum_{\mu=1}^{k} E_\mu^{(\xi)}$.

The pair energies $E_\mu^{(\xi)}$ should satisfy k coupled Bethe ansatz or Richardson-Gaudin equations:

$$1 = \sum_{j=1}^{p} \frac{1}{\eta_j - E_\mu^{(\xi)}} + \sum_{\nu \ne \mu} \frac{2}{E_\mu^{(\xi)} - E_\nu^{(\xi)}} \tag{4}$$

for $\mu = 1, 2, \cdots, k$. It is understood that the additional quantum number ξ in (2)-(4) is introduced to label the ξ-th set of roots $\{E_\mu^{(\xi)}\}$ of the equations (4).

To map the reduced BCS Hamiltonian (1) into a boson Hamiltonian, we first use the mapping that maps the fermion pair operators $\hat{k}_j$, $S_j^\pm$ into the corresponding real boson operators with

$$\hat{k}_j \mapsto n_j = b_j^\dagger b_j,\ S_j^+ \mapsto b_j^\dagger,\ \ S_j^- \mapsto b_j\ \ \forall\ j, \tag{5}$$

in which the images satisfy the usual commutation relations of boson operators: $[b_i, b_j^\dagger] = \delta_{ij}$, and $[b_i, b_j] = 0$. It is clear that this mapping is different from that based on group structure[14] because the images of $S_j^\pm$ no longer satisfy the commutation relations of the original SU(2) algebras. Furthermore, the mapping is unitary and number-conserving. We then seek a Bose Hamiltonian constructed from those boson images which should keep the wavefunction (3) consistent after the mapping. We found the one-body term in (1) does keep the same form after the mapping, while the pairing interaction term can not be mapped into one-body form, but with an additional non-Hermitian two-body interaction term, which is quite natural because the fermion pairing interaction like hard-core boson hopping can not be replaced by usual boson hopping. The final image of (1) after the mapping (5) is of the following form:

$$\hat{H}_{\rm Bose}/G = \sum_{j=1}^{p} (\eta_j - 1) n_j - \sum_{i \ne j} b_i^\dagger b_j + \sum_{i,j=1}^{p} n_j b_i^\dagger b_j. \tag{6}$$

To reveal the dynamics of the Hamiltonian (6), let us consider a more general form of (6) with

$$\hat{H}_{\rm BH} = \sum_j (2\epsilon_j - t - U) n_j - \sum_{i \ne j} (t - n_j U) b_i^\dagger b_j + U \sum_j n_j^2, \tag{7}$$

where $2\epsilon_j - t - U$ in the first term can be regarded as contribution from external potential or on-site disorder, the second term describes boson hopping among all sites with site dependent hopping parameter $t - n_j U$, and the third term is the on-site repulsion. Since the two-body interaction term usually contribute with the same order of magnitude as the one-body term, the on-site repulsion parameter may be set as $U = U_0/k$, where k is the total number of bosons, in which U_0 and t is of the same order of magnitude. Hence, the reduced BCS pairing Hamiltonian is mapped into a Bose-Hubbard model with site-dependent hopping parameter $t - U_0(n_j/k)$. Therefore, the more the bosons on the j-th level, the less the hopping strength of other bosons hopping onto j-th level if $t \ge U_0$. In the Bose Hamiltonian (6) with $t = U = G$, however, the condition $1 \ge n_j$ is no longer satisfied if $n_j \ne 0$ or 1, which means that the fermion pairing interaction looks extremely repulsive after the boson mapping (5).

To prove that (6) is indeed the exact boson image of (1), one can simply verify that the wavefunctions of (7), at least a part of them, can indeed be written as the boson image of (2) with

$$|k, \xi\rangle = B^+(E_1^{(\xi)}) B^+(E_2^{(\xi)}) \cdots B^+(E_k^{(\xi)}) |0\rangle_{\rm B}, \tag{8}$$

where $|0\rangle_{\rm B}$ is the corresponding boson vacuum, and

$$B^{+}(E_{\mu}^{(\xi)}) = \sum_{j=1}^{p} \frac{1}{2\epsilon_j/t - E_{\mu}^{(\xi)}} b_j^{\dagger}, \tag{9}$$

with the corresponding eigen-energies

$$E^{(\xi)} = t \sum_{\mu=1}^{k} E_{\mu}^{(\xi)} \tag{10}$$

and the Bethe ansatz equations

$$1 = \sum_{j=1}^{p} \frac{1}{2\epsilon_j/t - E_{\mu}^{(\xi)}} + \sum_{\nu \neq \mu} \frac{2(U/t)}{E_{\mu}^{(\xi)} - E_{\nu}^{(\xi)}} \tag{11}$$

for $\mu = 1, 2, \cdots, k$.

Though it is difficult to analyze the spectrum generated by (11) analytically, one can verify that the whole spectrum described by (10) obtained from solutions of (11) are real and complete with $(k+p-1)!/(k!(p-1)!)$ eigenvalues when $Uk/t \leq 1$. Simple examples for $p=3$, $k=2$, 3 are shown in Table 1.

Table 1. The pair energies or roots of (11) and eigen-energies given by (10) for $p=3$ and $k=2$ and 3 with $t=1$, $\epsilon_1 = 1.1$, $\epsilon_2 = 2.2$, $\epsilon_3 = 3.3$, and $U = 1/3$. The dimension is exactly equal to $(k+p-1)!/((p-1)!k!)$.

k	Dimension	Eigenvalues	Roots		
2	6	1.768	$E_1 = 0.884 - 0.792i$	$E_2 = 0.884 + 0.792i$	
		4.475	$E_1 = 0.935$	$E_2 = 3.540$	
		6.763	$E_1 = 0.735$	$E_2 = 6.028$	
		7.622	$E_1 = 3.811 - 0.358i$	$E_2 = 3.811 + 0.358i$	
		9.744	$E_1 = 3.770$	$E_2 = 5.974$	
		12.428	$E_1 = 6.214 - 0.258i$	$E_2 = 6.214 + 0.258i$	
3	10	3.930	$E_1 = 1.030$	$E_2 = 1.450 - 1.229i$	$E_3 = 1.450 + 1.229i$
		6.097	$E_1 = 3.343$	$E_2 = 1.377 - 0.589i$	$E_3 = 1.377 + 0.589i$
		8.237	$E_1 = 5.985$	$E_2 = 1.126 - 0.685i$	$E_3 = 1.126 + 0.685i$
		8.577	$E_1 = 1.227$	$E_2 = 3.675 - 0.425i$	$E_3 = 3.675 + 0.425i$
		10.704	$E_1 = 3.672$	$E_2 = 1.104$	$E_3 = 5.928$
		12.108	$E_1 = 3.896$	$E_2 = 4.106 - 0.568i$	$E_3 = 4.106 + 0.568i$
		13.302	$E_1 = 0.932$	$E_2 = 6.185 - 0.276i$	$E_3 = 6.185 + 0.276i$
		13.718	$E_1 = 5.822$	$E_2 = 3.948 - 0.302i$	$E_3 = 3.948 + 0.302i$
		16.142	$E_1 = 3.848$	$E_2 = 6.147 - 0.304i$	$E_3 = 6.147 + 0.304i$
		19.184	$E_1 = 6.286$	$E_2 = 6.449 - 0.357i$	$E_3 = 6.449 - 0.357i$

When $U = t = G$, with which (7) is reduced to (6) corresponding to the image of the reduced BCS pairing Hamiltonian (1). In this case, though it is not guaranteed that all eigenvalues of (7) are real, especially for large k cases, a part of them satisfying (11) consisting of $p!/((p-k)!k!)$ eigenvalues, which are the same as those given by (4), correspond exactly to those of the reduced BCS pairing Hamiltonian (1). There are $[(k+p-1)!/((p-1)!k!) - p!/((p-k)!k!)]$ more other eigenvalues which are not provided by (10)-(11). Hence, eigenvectors other than those corresponding to the eigenvalues given by (10) can not be written in the Bethe ansatz form (8).

Similar to the Dyson mapping[14] and, for example, the iterative boson expansion approach,[16] the resultant Bose Hamiltonian (6) is non-Hermitian. In addition, there will be spurious states involved in the boson space. Therefore, the Bose Hamiltonian should be projected onto the physical subspace. Let $\hat{P}$ be the projection operator, which can be expressed as

$$\hat{P} = \sum_{k\xi} |k,\xi)(\xi,k|, \tag{12}$$

where $|k,\xi)$ are normalized eigenvectors given by (8), and the sum runs over all possible values according to the number of solutions of (11) with $U = t = G$. Thus, it is clear that the projection operator $\hat{P}$ annihilates unphysical subspace.[14] It follows that the projected Bose Hamiltonian with

$$\tilde{H}_{\rm Bose} = \hat{P}\hat{H}_{\rm Bose}\hat{P} \tag{13}$$

is digonalizable under the physical subspace spanned by $\{|k,\xi)\}$ with results shown by (8)-(11). Hence, we obtain the Bose Hamiltonian $\tilde{H}_{\rm Bose}$ exactly equivalent to the reduced BCS Hamiltonian (1) in the physical boson subspace.

In summary, an exact boson mapping of the reduced BCS pairing Hamiltonian is obtained under the guidance of the Richardson-Gaudin exact solutions of the reduced BCS pairing model. Though non-Hermitian, all solutions of the resultant Bose-Hubbard Hamiltonian are real and provided by the Richardson-Gaudin type wavefunctions and the corresponding Bethe ansatz equations when $Uk/t < 1$. The physical Bose image of the reduced BCS pairing Hamiltonian is obtained by the projection method which is exactly equivalent to the reduced BCS Hamiltonian (1) in the physical boson subspace. Because what we have studied is based on deformed shell model like basis, the boson operators $\{b_j, b_j^\dagger\}$ do not conserve angular momentum which must be restored by angular momentum projection as mentioned in [17]. In the Nilsson mean-filed, for example, the boson operator $b_i^\dagger$ can be rewritten in terms of spherical pairs as

$$b_i^\dagger = x_0^i s_0^+ + x_2^i d_0^\dagger + x_4^i g_0^\dagger + \cdots, \tag{14}$$

where x_L^i are transformation coefficients between the i-th Nilsson level and the spherical basis, and s_0^+, $d_0^\dagger$, etc. are boson operators with $L = 0, 2, \cdots$, and $M_L = 0$. After the angular momentum projection, one can better understand the intimate links between mean-field plus pairing models and the interacting boson model. Further study about this problem will be carried out in the near future.

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References

[1] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[2] M. Randeria, J. M. Duan, L.Y. Shieh, Phys. Rev. Lett. **62**, 981 (1989).

[3] D. J. Dean, Rev. Mod. Phys. **75**, 607 (2003).

[4] D. C. Ralph, C. T. Black, and M. Tinkham, Phys. Rev. Lett. **78**, 4087 (1996).

[5] L. Amico, A. Di Lorenzo, and A. Osterloh, Phys. Rev. Lett. **86**, 5759 (2001).

[6] D. W. Cooper, J. S. Batchlder, M. A. Taubenblatt, J. Coll. Int. Sci. **144**, 201 (1991).

[7] P. Ring and P. Schuck, The Nuclear Many-Body Problem (Springer-Verlag, Berlin, 1980).

[8] For example, J. Dukelsky, G. G. Dussel, J. G. Hirsch and P. Schuck, Nucl. Phys. A **714**, 63 (2003), and references therein.

[9] R. W. Richardson, Phys Lett. **3**, 277 (1963).

[10] R. W. Richardson,J. Math. Phys. **6**, 1034 (1965).

[11] M. Gaudin, J. Physique **37**, 1087(1976).

[12] J. Dukelsky, C. Esebbag, and S. Pittel, Phys. Rev. Lett. **88**, 062501 (2002).

[13] S. Rombouts, D. Van Neck, and J. Dukelsky, Phys. Rev. C **69**, 061303(R) (2004).

[14] A. Klein, E. R. Marshalek, Rev. Mod. Phys. **63**, 375(1991), and references therein.

[15] F. Iachello and A. Arima, The Interacting Boson Model (Cambridge University Press, Cambridge, 1987).

[16] J. Dukelsky and S. Pittel, Phys. Rev. C **39**, 2001 (1989).

[17] F. Iachello and I. Talmi, Rev. Mod. Phys. **59**, 339 (1987), and references therein.